RESEARCH STATEMENT

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My primary research interests lie in the area of number theory and arithmetic geometry. Specifically, I am interested in modular forms, p-adic L-functions, and Iwasawa theory. The main result of my Ph.D. thesis will be a generalization of Ohta's proof of the Iwasawa main conjecture over $\mathbb{Q}$. While simple and elegant, Ohta's proof requires several restrictive hypotheses that one would like to remove. The motivation behind removing these hypotheses stems from the role Ohta's work plays in the recent conjectures of Sharifi [14] relating cohomology groups of modular curves to ideal class groups, and the partial proof of these conjectures by Fukaya and Kato [2]. We hope to be able to remove the corresponding hypotheses in the work of Sharifi and Fukaya-Kato by addressing the underlying obstructions in the context of Ohta's proof of the Iwasawa main conjecture.

1 BACKGROUND

In this section we give a brief introduction to the Iwasawa main conjecture over $\mathbb{Q}$ and introduce much of the notation that will be used throughout.

There are many "main conjectures" in Iwasawa theory, but all have the same distinguishing characteristic, namely, they give an analytic description of arithmetic invariants. In the case of the main conjecture over $\mathbb{Q}$, an invariant of ideal class groups is described in terms of p-adic L-functions.

Let's begin by considering the arithmetic side of this story. Fix an odd prime p and let θ, ψ be p-adic Dirichlet characters whose product is even (i.e. $(\theta\psi)(-1) = 1$). Furthermore, if θ and ψ are defined modulo M_θ and M_ψ, respectively, let us assume that $M_\theta M_\psi = N$ or Np, where N is some integer prime to p. Considering θ and ψ as Galois characters, we define F to be the field corresponding to $\ker(\theta\omega) \cap \ker(\psi)$, where ω is the Teichmüller character. Let F_∞ denote the cyclotomic $\mathbb{Z}_p$-extension of F, with F_n being the unique subextension satisfying $\mathrm{Gal}(F_n/F) \cong \mathbb{Z}/p^n\mathbb{Z}$. We denote the Sylow p-subgroup of the ideal class group of F_n by X_n, and set

$$X := \varprojlim_n X_n$$

where the projective limit is taken with respect to norm maps. It can be shown that there is a natural action of $\Delta \times \Gamma := \mathrm{Gal}(F/\mathbb{Q}) \times \mathrm{Gal}(F_\infty/F)$ on X, and with this action in mind, we set

$$X_\xi := X \otimes_{\mathbb{Z}_p[\Delta]} \mathbb{Z}_p[\xi]$$

where $\xi = (\theta\psi^{-1}\omega)^{-1}$ and $\mathbb{Z}_p[\xi]$ is the subring of $\overline{\mathbb{Q}}_p$ (the algebraic closure of $\mathbb{Q}_p$) generated over $\mathbb{Z}_p$ by the values of ξ. The group Δ then acts on X_ξ via the character ξ and one can show that X_ξ is a finitely generated, torsion $\mathbb{Z}_p[\xi][\![\Gamma]\!]$-module. Thanks to Serre [15] we have structure theorem for such modules. Let $\Lambda_\xi = \mathbb{Z}_p[\xi][\![T]\!]$ (i.e. power series in T with coefficients in $\mathbb{Z}_p[\xi]$). One can identify $\mathbb{Z}_p[\xi][\![\Gamma]\!]$ with Λ_ξ via the map $\gamma \mapsto 1+T$, where γ is a topological generator of Γ (for example, take γ to be the image of 1 under the isomorphism $\Gamma \cong \mathbb{Z}_p$). Then there exists a Λ_ξ-module homomorphism

$$X_\xi \to \Lambda_\xi/(f_1) \oplus \cdots \oplus \Lambda_\xi/(f_r)$$

with finite kernel and cokernel, such that the $f_i \in \Lambda_\xi$ are non-zero divisors uniquely determined by X_ξ. From this structure theorem we are able to define the most important invariant associated to X_ξ, its characteristic ideal, which is defined by

$$\mathrm{Char}(X_\xi) = (f_1 \cdots f_r).$$

This ideal is the invariant that will appear in the Iwasawa main conjecture over $\mathbb{Q}$.

On the analytic side, Iwasawa [6] has shown that if the conductor of ξ is not divisible by p^2, there exists an element $G(T, \xi^{-1}\omega) \in \Lambda_\xi$ such that

$$G((1+p)^s - 1, \xi^{-1}\omega) = L_p(s, \xi^{-1}\omega)$$

for all $s \in \mathbb{Z}_p$, where $L_p(s, \xi^{-1}\omega)$ is the p-adic L-function attached to $\xi^{-1}\omega$. The Iwasawa main conjecture over $\mathbb{Q}$ then states that

$$\mathrm{Char}(X_\xi) = G(T, \xi^{-1}\omega)\Lambda_\xi.$$

Despite its name, the main conjecture is a theorem. It was first proven by Mazur and Wiles [8], and has subsequently been generalized by Wiles [17] and Rubin [13]. More recently, Ohta ([10], [11], [12]) has given a simple proof of the main conjecture in the flavor of Mazur-Wiles and Wiles. However, as was mentioned earlier, this proof comes with some restrictive hypotheses. Specifically, Ohta requires

(i) $p \geqslant 5$

(ii) $p \nmid \phi(N)$ (ϕ is Euler's totient function)

(iii) θ, ψ are primitive

(iv) θ, ψ are non-exceptional, (i.e. $(\theta\psi^{-1}\omega)(p) \neq 1$.)

In my thesis I will generalize Ohta's proof by removing hypotheses (ii)-(iv).

While I have made many non-trivial modifications and several simplifications to Ohta's proof, in the next two sections I would like to focus on the main results of my thesis pertaining to conditions (iii) and (iv).

Throughout this section let us assume that $p \geqslant 5$, with θ and ψ as in the previous section with the additional condition that $(\theta, \psi) \neq (\omega^{-2}, 1)$. Recall that θ and ψ are arbitrary (i.e. possibly imprimitive or exceptional). We will denote the primitive character associated to a given character χ by χ_{prim} and its conductor by f_χ.

2.1 *Cuspidal Hecke algebra modulo Eisenstein ideal*

In this section we will address the fundamental problem that arises when the pair (θ, ψ) is exceptional.

Set $\Lambda := \mathbb{Z}_p[\theta, \psi][\![T]\!]$, where $\mathbb{Z}_p[\theta, \psi]$ is the subring of $\overline{\mathbb{Q}}_p$ generated over $\mathbb{Z}_p$ by the values of θ and ψ. We begin by defining ordinary Λ-adic modular forms as introduced by Hida [4] and Wiles [16]. We say that a formal q-expansion $F = \sum_{n=0}^{\infty} a_n(F)(T)\, q^n \in \Lambda[\![q]\!]$ is an ordinary Λ-adic modular form of level N, if for all integers $k \geqslant 2$ and p-power roots of unity ζ,

$$F_{\zeta,k} \;:=\; \sum_{n=0}^{\infty} a_n(F)\Big(\zeta \cdot (1+p)^{k-2} - 1\Big)\; q^n \tag{1}$$

is an ordinary modular form of weight k and level Np^{r+1}, where p^r is the order of ζ. Here, ordinary means that the action of the Hecke operator T_p on the form is invertible. If $F_{\zeta,k}$ is a cusp form for all k and ζ, we say that F is an ordinary Λ-adic cusp form. Let M_Λ (resp., S_Λ) denote the space of ordinary Λ-adic modular forms (resp., cusp forms) of level N. Of particular interest are the Λ-adic Eisenstein series $\mathcal{E}(\theta, \psi; t) \in M_\Lambda$ for integers $t \geqslant 1$ prime to p, due to the fact that

$$a_0(\mathcal{E}(\theta, \psi; t)) \;=\; \frac{\psi(0)}{2} G(u^{-1}(1+T)^{-1} - 1, \theta\omega^{-2}).$$

As with classical modular forms, for all integers $n \geqslant 1$ we can define Hecke operators T_n, and their adjoint T_n^*, acting on M_Λ and S_Λ. Furthermore, the action of these operators agrees with that of the classical operators under the specialization map (1). We let $\mathfrak{H}$ (resp., $\mathfrak{h}$) denote the Λ-subalgebra of $\mathrm{End}_\Lambda(M_\Lambda)$ (resp., $\mathrm{End}_\Lambda(S_\Lambda)$) generated by the operators T_n for all $n \geqslant 1$. We define $\mathfrak{H}^*$ and $\mathfrak{h}^*$ analogously with respect to the adjoint operators T_n^*. The Eisenstein ideal $I \subset \mathfrak{h}$, is defined to be the image of $\mathrm{Ann}_{\mathfrak{H}}(\mathcal{E}(\theta, \psi))$ under the natural restriction map $\mathfrak{H} \twoheadrightarrow \mathfrak{h}$. We define $I^* \subset \mathfrak{h}^*$ to be the image of I under the Λ-algebra isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$ defined by $T_n \mapsto T_n^*$.

In [10] and [12], Ohta constructs an exact sequence of Hecke modules

$$0 \longrightarrow S_\Lambda \xrightarrow{\text{inc}} M_\Lambda \xrightarrow{\mathrm{Res}_\Lambda} \mathcal{C} \longrightarrow 0,$$

where $\mathcal{C}$ is the ordinary Λ-adic cuspidal group with respect to the modular curves $X_1(Np^r)$ for $r \geqslant 1$. Here, the first map is the natural inclusion and the second is what Ohta refers to as the Λ-adic residue map. In my thesis I prove the following proposition.

Proposition 2.1. *Let* $t \geqslant 1$ *be prime to* p, *then*

$$\mathrm{Res}_\Lambda\big(\mathcal{E}(\theta_{\mathrm{prim}}, \psi_{\mathrm{prim}}; t)\big) \;=\; U \cdot A \cdot G(u^{-1}(1+T)^{-1} - 1; \xi^{-1}\omega) \cdot \mathfrak{c}$$

for some $\mathfrak{c} \in \mathcal{C}$, *where* $U \in \Lambda^\times$ *and*

$$A = \prod_{\substack{\ell | f_\theta f_\psi \\ \ell \nmid f_{\theta\psi^{-1}}}} \Big((1+T)^{s(\ell)} - (\theta^{-1}\psi)(\ell) \cdot \ell^{-2}\Big)$$

with $s(\ell) \in \mathbb{Z}_p$ *is defined by* $u^{s(\ell)} = \ell\omega(\ell)^{-1}$.

We remark that in the course of the proving Proposition 2.1, U and $\mathfrak{c}$ are explicitly determined. The above result was proven by Ohta in [12] for $t = 1$ and $(\theta_{\mathrm{prim}}, \psi_{\mathrm{prim}})$ non-exceptional. However, his methods cannot be generalized to the exceptional case.

The primary utility of Proposition 2.1 is in determining the image of $\mathcal{E}(\theta, \psi)$ under the Λ-adic residue map for arbitrary θ and ψ (i.e. imprimitive or exceptional). We do so by noting that for such θ and ψ, the series $\mathcal{E}(\theta, \psi)$ can be written as a Λ-linear combination of the series $\mathcal{E}(\theta_{\mathrm{prim}}, \psi_{\mathrm{prim}}; t)$ for various t. This in turn allows us to determine congruences between Λ-adic cusp forms and Λ-adic Eisenstein series. Specifically, it implies the existence of a Λ-adic cusp form F satisfying $F \equiv \mathcal{E}(\theta, \psi) \pmod{(A)}$. Using this congruence I was able to give a simple proof of the following important corollary.

Corollary 2.2. *We have a surjection of* Λ*-algebras,*

$$\mathfrak{h}/I \twoheadrightarrow \Lambda/(A).$$

This was first proven by Wiles [17] in the case when $\psi = 1$ and $(\theta, 1)$ is non-exceptional. In [12], Ohta removes the triviality condition on ψ. Using Katz's p-adic Modular forms, Emerton [1] has shown that the above is an isomorphism when $\psi = 1$ and θ is a power of the Teichmüller character. We remark that as a consequence of the Iwasawa main conjecture over $\mathbb{Q}$ one can show $\mathfrak{h}/I \cong \Lambda/(A)$. Moreover, there is currently no proof of this fact that does not employ the main conjecture.

2.2 *Constructing extensions with prescribed ramification*

In this section we describe the construction of an abelian pro-p extension L of F_∞, and the ramification in L/F_∞ when we allow our characters to be imprimitive.

We begin by introducing an object fundamental to both Ohta's proof and Sharifi's conjectures; namely, the p-adic Eichler-Shimura cohomology group of level N, which is defined by

$$H \;:=\; \varprojlim_r H^1_{\acute{e}t}(X_1(Np^r) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Z}_p)^{\mathrm{ord}} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[\theta, \psi]$$

where $X_1(Np^r)$ is the canonical model over $\mathbb{Q}$ in which the cusp at infinity is $\mathbb{Q}$-rational. There is a natural action of $\mathfrak{h}^*[G_\mathbb{Q}]$ on H and it is well known that $H \otimes_\Lambda Q(\Lambda)$ is a free $\mathfrak{h}^* \otimes_\Lambda Q(\Lambda)$-module of rank two, where $Q(\Lambda)$ denotes the quotient field of Λ. From this fact we see that H gives us a two-dimensional Galois representation

$$\rho : G_\mathbb{Q} \to GL_2(\mathfrak{h}^* \otimes_\Lambda Q(\Lambda)).$$

Let $\tilde{\rho}$ denote the reduction of ρ modulo I^* and let L be the fixed field of $\ker(\tilde{\rho})$. It can be shown that L is a pro-p abelian extension of F_∞, and that we have an isomorphism

$$\mathrm{Gal}(L/F_\infty) \to B/I^*B,$$

where B is a faithful $\mathfrak{h}^*$-submodule of $\mathfrak{h}^* \otimes_\Lambda Q(\Lambda)$. By considering the natural action of $\Delta \times \Gamma$ on B/I^*B, one can show that $\mathrm{Gal}(L/F_\infty)$ is a Λ_ξ-module on which Δ acts via the character ξ. Moreover, employing the surjection of Proposition 2.1, the faithfulness of B, and several basic results on Fitting ideals, we get the following inclusion:

$$\mathrm{Char}(\mathrm{Gal}(L/F_\infty)) \subseteq \left(\prod_{\substack{\ell \mid f_\theta f_\psi \\ \ell \nmid f_{\theta\psi^{-1}}}} b_\ell(T) \right) \cdot G(T, \xi^{-1}\omega),$$

where $b_\ell(T) := (1+T)^{s(\ell)} - \xi(\ell)\ell$. On the other hand, Ohta [12] shows that we also have the inclusion

$$\left(\prod_{\substack{\text{primes } \ell \text{ that} \\ \text{ramify in } L/F_\infty}} b_\ell(T) \right) \cdot \mathrm{Char}(\mathrm{Gal}(L^{un}/F_\infty)) \subseteq \mathrm{Char}(\mathrm{Gal}(L/F_\infty)),$$

where L^{un} denotes the maximal unramified subextension of L/F_∞. By a well known consequence of the analytic class number formula, to prove the Iwasawa main conjecture over $\mathbb{Q}$ it suffices to show $\mathrm{Char}(\mathrm{Gal}(L^{un}/F_\infty)) \subseteq G(T, \xi^{-1}\omega)$. From the above inclusions we see that this is equivalent to showing that the only primes ℓ that ramify in L/F_∞ are those that divide $f_\theta f_\psi$ but do not divide $f_{\theta\psi^{-1}}$.

Using Igusa's theorem, one can show that the primes $\ell \nmid N$ are unramified in L/F_∞, and by employing an argument of Mazur-Wiles [8] one can show that the primes ℓ dividing $f_{\theta\psi^{-1}}$ are unramified as well. In my thesis I expect to show that the extension L/F_∞ is also unramified at the primes dividing N that do not divide $f_\theta f_\psi$. I will do so by studying the image of $\tilde{\rho}$ under the degeneracy maps from the Eichler-Shimura groups of level N to those of level $f_\theta f_\psi$, effectively showing that the representation $\tilde{\rho}$ is "old" in the sense that it comes from the lower level $f_\theta f_\psi$.

3 FUTURE RESEARCH

3.1 *The case when* $p = 2$ *or* 3

A natural question to tackle next would be to remove the restriction on the prime p. The fundamental obstruction in this case is the fact that the modular curve $X_1(Np^r)$ is not a fine moduli space for $Np^r < 5$. There are several ways to attack this question, but perhaps the path of least resistance is to determine how the various objects in our proof are affected by requiring $r \geqslant 3$. I have already made some progress on this question in my earlier investigations into Ohta's work, and look forward to revisiting this soon.

3.2 *Hecke algebra modulo Eisenstein Ideal revisited*

As mentioned at the end of Section 2.1, one can use the Iwasawa main conjecture over $\mathbb{Q}$ to conclude that the surjection of $\mathfrak{h}/I \twoheadrightarrow \Lambda/(A)$ of Proposition 2.1 is also injective. It would be nice to have such a proof that did not require the full force of the main conjecture (i.e. a more general version of Emerton's argument [1]). This is fundamentally a question of duality between Hecke algebras and Λ-adic modular forms. Let me briefly explain why. Set $\mathcal{P} := \{F \in M_\Lambda : \mathrm{Res}_\Lambda(F) \in \Lambda(\mathfrak{c})\}$, where $\mathfrak{c}$ is as in Proposition 2.1, and let $\mathfrak{H}_\mathcal{P}$ be the restriction of $\mathfrak{H}$ to $\mathcal{P}$. Using an argument involving congruence modules one can show that the desired isomorphism is equivalent to showing $\mathfrak{H}_\mathcal{P} \cong \mathrm{Hom}_\Lambda(\mathcal{P}, \Lambda)$ via the map $H \mapsto \varphi_H$ with $\varphi_H(F) = a_1(F|H)$. In [1], Emerton proves a similar duality using Katz's Key Lemma [7] in a fundamental way. Any proof of the isomorphism $\mathfrak{h}/I \cong \Lambda/(A)$ will require a similar result for the submodule $\mathcal{P}$.

3.3 *Sharifi's conjectures*

As mentioned in the introduction, Sharifi [14] has made some remarkable conjectures relating the p-adic Eichler-Shimura cohomology groups H and the projective limit of ideal class groups X. Specifically, he constructs maps $\varpi : H^-/IH^- \to X^-$ and $\Upsilon : X^- \to H^-/IH^-$ (where the superscript "$^-$" indicates that complex conjugation acts by -1) and conjectures that

$$\varpi \circ \Upsilon = 1 = \Upsilon \circ \varpi.$$

One can show that the Iwasawa main conjecture over $\mathbb{Q}$ would be a consequence of this deeper relationship [3]. However, in order to construct the map Υ, the hypotheses (i)-(iv) are assumed. I would like to show that Sharifi's map Υ can be constructed without these hypotheses by employing the techniques used to circumvent the corresponding obstructions in Ohta's proof of the Iwasawa main conjecture over $\mathbb{Q}$.

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